THE MOTION OF A PARTICLE IN A NONUNIFORM MOVING VISCOUS MEDIUM

A. T. Litvinov

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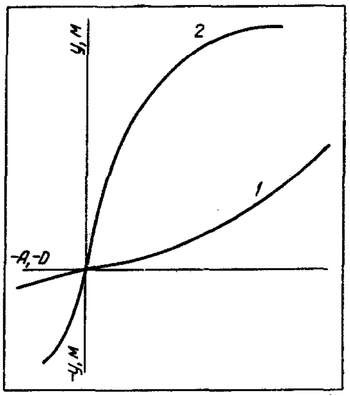
The motion of particles is examined in a nonuniform moving viscous medium in the regions Re < 1 and 1 < Re ≤ 300. Solutions are presented for the differential equations which make it possible to calculate the trajectories of particle motion in a nonuniform moving viscous medium.

The differential equation of motion for a particle in a nonuniform moving viscous medium is

$$m\frac{d}{dt}\mathbf{V}_p = 0.5\psi S\rho_1(\mathbf{V}_m-\mathbf{V}_p)|\mathbf{V}_m-\mathbf{V}_p| + m_1\frac{d}{dt}(\mathbf{V}_m-\mathbf{V}_p) + 1.5d^2\sqrt{\pi\rho_1\mu}\int_{t_0}^{t_1}\frac{\frac{d}{dt_2}(\mathbf{V}_m-\mathbf{V}_p)}{\sqrt{t_0-t_2}}\,dt_2 + \mathbf{F}_e. \quad (1)$$

The term on the left-hand side of (1) defines the force needed to produce the acceleration of the moving particle. The first term in the right-hand side defines the force of resistance to motion, resulting from the viscosity of the medium; the second term defines the force caused by the acceleration of the medium as the magnitude of the velocity is altered; the third term defines the force needed to set the apparent mass into motion (for spherical particles the apparent mass is equal to half the mass displaced by the medium sphere); the fourth, integral term takes into consideration the force expended to overcome the additional resistance of the medium, resulting from the change in the velocity of particle and medium motion; the fifth term is the external force applied to the particle.



Paths of entrainment (deceleration) of the particles by the medium as functions of parameters L (curve 1) and D (curve 2).

The nonuniform motion of a particle in a resting medium has been considered in [1, 2], while the effect of the integral term on the motion of the particle has been calculated in [3]. However, the author's evaluation of the extent of influence exerted by the integral term is applicable to particles with a density ratio <0.15.

The nonuniform motion of particles in a nonuniform moving medium was examined in [4–7] and the extent of the influence of the integral term was evaluated for the nonuniform motion of a medium and particle, it also having been demonstrated that the integral term may exert considerable influence on the coefficient of hydrodynamic resistance ψ.

The nonuniform motion of particles in a resting medium with a coefficient of hydrodynamic resistance partially taking into consideration the inertial terms has been considered in [8–12].

Analysis of the integral term in (1) shows that when $t_0 = t_2$ the integrand becomes infinite. However, the increase in the integral term is proportional to $\lim_{\tau\to 0}\tau^{-0.5}$, and it therefore remains a finite quantity. This indicates that with a rapid change in particle or medium motion (a large value for the force of acceleration) the value of the instantaneous coefficient of hydrodynamic resistance may considerably exceed its magnitude for steady-state velocity.

To solve (1) in the region Re < 1 ($\psi = 24/\mathrm{Re}$) we select a system of coordinates moving together with the particle. Equation (1) is then transformed (without consideration of the external force) to

$$\frac{\pi d^3}{6}\rho_2\frac{d}{dt}\mathbf{V}_p = 3\pi\mu d(\mathbf{V}_m-\mathbf{V}_p)|\mathbf{V}_m-\mathbf{V}_p| + \frac{\pi d^3}{6}\rho_1\frac{d}{dt}\mathbf{V}_m + \frac{\pi d^3}{12}\rho_1\frac{d}{dt}(\mathbf{V}_m-\mathbf{V}_p) + 1.5d^2\sqrt{\pi\rho_1\mu}\int_{t_0}^{t_1}\frac{\frac{d}{dt_2}(\mathbf{V}_m-\mathbf{V}_p)}{\sqrt{t_0-t_2}}\,dt_2. \quad (2)$$

It was demonstrated in [6] that the second term in (2) can be expressed in terms of the pressure gradient without violating generality:

$$-\frac{\partial \mathbf{P}}{\partial x_i} = \rho_1\left[\frac{\partial(\mathbf{V}_m)_i}{\partial t} + (\mathbf{V}_m)_k\frac{\partial(\mathbf{V}_m)_i}{\partial x_k}\right] - \nu\frac{\partial^2(\mathbf{V}_m)_i}{\partial x_i^2}. \quad (3)$$

Equation (2) for the i-th component of medium velocity $(\mathbf{V}_m)_i$ is then transformed to

$$\frac{d}{dt}(\mathbf{V}_p)_i = \alpha_1[(\mathbf{V}_m)_i-(\mathbf{V}_p)_i] + \beta_1\left[\frac{\partial}{\partial t}(\mathbf{V}_m)_i + (\mathbf{V}_m)_k\frac{\partial(\mathbf{V}_m)_i}{\partial x_k}\right] - \gamma_1\frac{\partial^2(\mathbf{V}_m)_i}{\partial x_i^2} + \zeta_1\frac{d}{dt}[(\mathbf{V}_m)_i -$$

$$-(V_p)_i]+\varphi_1\int_{t_0}^{t_1}\frac{\frac{d}{dt_2}[(V_m)_i-(V_p)_i]}{\sqrt{t_0-t_2}}dt_2, \quad (4)$$

where $\alpha_1 = 18\mu(d^2\rho_2)^{-1}$; $\beta_1 = \rho_1/\rho_2$; $\gamma_1 = \mu/\rho_2$; $\zeta_1 = 0.5\beta_1$; $\varphi_1 = 9(\pi\rho_1\mu)^{1/2}(\pi d\rho_2)^{-1}$.

Having substituted into (4) the value of

$$\frac{\partial}{\partial t}(V_m)_i+(V_m)_k\frac{\partial(V_m)_i}{\partial x_k}=$$

$$=\frac{d}{dt}(V_m)_i+[(V_m)_k-(V_p)_k]\frac{\partial}{\partial x_k}(V_m)_i,$$

after appropriate transformations we obtain

$$\frac{d}{dt}(V_p)_i=b\left[\frac{d}{dt}(V_m)_i-\frac{2}{3}\nu\frac{\partial^2}{\partial x_i^2}(V_m)_i\right]+$$

$$+B\left\{\frac{18\mu}{d^2}[(V_m)_i-(V_p)_i]+\rho_1[(V_m)_k-\right.$$

$$\left.-(V_p)_k]\frac{\partial}{\partial x_k}(V_m)_i\right\}+C\int_{t_0}^{t_1}\frac{\frac{d}{dt_2}[(V_m)_i-(V_p)_i]}{\sqrt{t_0-t_2}}dt_2, \quad (5)$$

where

$$b=1.5B\rho_1;\quad B=(\rho_2+0.5\rho_1)^{-1};$$

$$C=9Bd^{-1}\sqrt{\frac{\rho_1\mu}{\pi}}.$$

Analysis of (5) shows that the nonlinear terms can be neglected (without great error) when

$$\rho_1\frac{\partial}{\partial x_k}|V_m|\ll\frac{18\mu^2}{d^2}\quad\text{or}\quad\frac{d^2}{\nu}\frac{\partial}{\partial x_k}|V_m|\ll 1. \quad (6)$$

Condition (6) is easily satisfied for 10^{-6}–10^{-4}-m particles. Thus, for example, for particles with $d = 10^{-6}$ m we have $7\cdot 10^{-12}(\partial/\partial x_k)|V_m| \ll 1$. Thus even for large values of the velocity gradient $(\partial/\partial x_k)\times|V_m| \sim 10^7$ the value of $(d^2/\nu)(\partial/\partial x_k)|V_m| \ll 1$.

The effect of viscosity can be neglected when

$$|V_p|_i\frac{\partial}{\partial x_k}|V_m|\gg\nu\frac{\partial^2}{\partial x_k^2}|V_m|_i. \quad (7)$$

On the basis of (6) we can rewrite (7) as

$$\frac{(V_p)_i}{d^2}\frac{1}{\partial^2|V_m|_i/\partial x_k^2}\gg 1. \quad (8)$$

Analysis of (8) shows that for particles of the above-indicated dimensions (1–100 μm) the condition in (8) is satisfied without great error in engineering calculations.

Dropping the nonlinear terms and the subscripts of the i-th component for the particle and medium velocities, we transform (5) to

$$\frac{dV_p}{dt}+aV_p=aV_m+b\frac{dV_m}{dt}+$$

$$+C\int_{t_0}^{t_1}\frac{\frac{d}{dt_2}(V_m-V_p)}{\sqrt{t_0-t_2}}dt_2, \quad (9)$$

where $a = 36\mu/d^2(2\rho_2+\rho_1)$.

We rewrite (9) as

$$y''+ay'=f(t), \quad (10)$$

where

$$f(t)=b\frac{dV_m}{dt}+aV_m+C\int_{t_0}^{t_2}\frac{\frac{d}{dt_2}(V_m-V_p)}{\sqrt{t_0-t_2}}dt_2.$$

The solution of the differential equation (10) for the initial conditions $t_1 = t_0 = 0$; $y' = V_i$; $y = 0$ is

$$y=\frac{a}{V_i}(1-e^{-at})+$$

$$+\frac{2}{a}\int_{t_0}^{t_1}f(x)\exp[0.5a(x-t)]\,\mathrm{sh}\frac{a}{2}(t-x]\,dx. \quad (11)$$

For the solution of the integral term in (11) we employ the method of successive integration. Let us agree that for each time segment (integration interval) the medium velocity V_m is a constant equal to its value at the beginning of the interval. Knowing the initial conditions and integrating successively, we determine the trajectory of particle motion for nonuniform particle motion. Depending on the magnitude of the selected integration interval we attain the desired degree of calculation accuracy.

Assuming $f(x) = L$, we transform (11) to

$$y=\zeta d^2Lt+\frac{a^3-LV_i}{a^2V_i}(1-e^{-at}), \quad (12)$$

where $\zeta = (2\rho_2+\rho_1)/36\mu$.

With particle motion in the region Re ≤ 300 we write (2) in the form

$$\frac{\pi d^3}{6}\rho_2\frac{dV_p}{dt}=\left(A+\frac{B}{Re}\right)\frac{\pi d^2}{8}\rho_1(V_m-V_p)|V_m-$$

$$-V_p|+\frac{\pi d^3}{6}\rho_1\frac{dV_m}{dt}+\frac{\pi d^3}{12}\rho_1\frac{d}{dt}(V_m-V_p)+$$

$$+1.5d^2\sqrt{\pi\rho_1\mu}\int_{t_0}^{t_1}\frac{\frac{d}{dt_2}(V_m-V_p)}{\sqrt{t_0-t_2}}dt_2, \quad (13)$$

where A and B are constants equal for the region of motion Re ≤ 300, respectively, to 0.12 and 37 (for more exact calculations, the values of the constants are given in [13]). It is demonstrated in [13, 14] that the calculated values of the coefficient $\psi = A + B/Re$ are in satisfactory agreement with experimental data for decelerated particle motion in a resting medium, i.e., the coefficient of hydrodynamic resistance ψ takes into consideration the inertial terms (the third and fourth terms in (13)). We therefore transform (13) to

$$\frac{\pi d^3}{6}\rho_2\frac{d\mathbf{V}_p}{dt} = \left(A = \frac{B}{Re}\right) \times$$

$$\times (\mathbf{V}_m - \mathbf{V}_p)\,|\,\mathbf{V}_m - \mathbf{V}_p\,|\,\rho_1\frac{\pi d^2}{8} + \frac{d\mathbf{V}_m}{dt}\frac{\pi d^3}{6}\rho_1. \qquad (14)$$

After appropriate transformations of (14) we obtain

$$\frac{d\mathbf{V}_0}{dt} = \alpha V_0^2 + \beta V_0 + (\beta - 1)\frac{d\mathbf{V}_m}{dt}, \qquad (15)$$

where $\alpha = 0.75A\rho_1/d\rho_2$; $\beta = 0.75B\mu/d^2\rho_2$.

For successive integration we assume $(\beta - 1)\cdot(dV_m/dt) = D$. Then, multiplying both parts of (15) by dy, after appropriate transformations, we write

$$dy = \frac{d\mathbf{V}_0(\mathbf{V}_m - \mathbf{V}_0)}{\alpha\mathbf{V}_0^2 + \beta\mathbf{V}_0 + D}. \qquad (16)$$

Having integrated (16) within limits from $y_1 = 0$ to $y_2 = y$, and the relative particle velocity from $\mathbf{V}_0 = \mathbf{V}_i$ to V_0, we obtain the following solutions:

a) when $4\alpha D > \beta^2$

$$y_1 = \int_{\mathbf{V}_i}^{\mathbf{V}_0}\frac{d\mathbf{V}_0(\mathbf{V}_m - \mathbf{V}_0)}{\alpha\mathbf{V}_0^2 + \beta V_0 + D} =$$

$$= \mathbf{V}_m\int_{V_i}^{V_0}\frac{d\mathbf{V}_0}{\alpha V_0^2 + \beta V_0 + D} - \frac{1}{2\alpha}\ln M_2 +$$

$$+ \frac{\beta}{2\alpha}\int_{\mathbf{V}_i}^{\mathbf{V}_0}\frac{d\mathbf{V}_0}{\alpha\mathbf{V}_0^2 + \beta\mathbf{V}_0 + D} = \frac{2\mathbf{V}_m^2 + E\mathbf{V}_m}{M}\times$$

$$\times \operatorname{arctg}\frac{2\alpha M(\mathbf{V}_0 - \mathbf{V}_i)}{M^2 + (2\alpha\mathbf{V}_0 + \beta)(2\alpha\mathbf{V}_i + \beta)} - \frac{1}{2\alpha}\ln M_2, \quad (17)$$

where $M_1 = (V_m + 0.5E)[\alpha(q_2 - q_1)]^{-1}$; $q_{1,2} = 2D(\beta \pm N)^{-1}$; $N = (\beta^2 - 4\alpha D)^{1/2}$;

c) when $\beta^2 > 4\alpha D$ and $(2\alpha\mathbf{V}_0 + \beta)^2 < (\beta^2 - 4\alpha D)$

$$y_2 = \int_{\mathbf{V}_i}^{\mathbf{V}_0}\frac{dV_0(\mathbf{V}_m - \mathbf{V}_0)}{\alpha V_0^2 + \beta V_0 + D} =$$

$$= M_1\ln\frac{(\mathbf{V}_0 + q_1)(\mathbf{V}_i + q_2)}{(\mathbf{V}_0 + q_2)(\mathbf{V}_i + q_1)} - \frac{1}{2\alpha}\ln M_2, \qquad (18)$$

when $M_1 = (V_m + 0.5E)\,[\alpha(q_2 - q_1)]^{-1}$; $q_{1,2} = 2D(\beta \pm N)^{-1}$; $N = (\beta^2 - 4\alpha D)^{1/2}$;

c) when $\beta^2 > 4\alpha D$ and $(2\alpha V_0 + \beta)^2 < (\beta^2 - 4\alpha D)$

$$y_3 = \int_{\mathbf{V}_i}^{\mathbf{V}_0}\frac{d\mathbf{V}_0(V_m - V_0)}{\alpha\mathbf{V}_0^2 + \beta\mathbf{V}_0 + D} =$$

$$= \omega\,\operatorname{arth}\frac{2\alpha N(\mathbf{V}_0 - \mathbf{V}_i)}{N^2 - (2\alpha\mathbf{V}_0 + \beta)(2\alpha\mathbf{V}_i + \beta)} -$$

$$- \frac{1}{2\alpha}\ln M_2, \qquad (19)$$

where $\omega = (2V_m + E)/N^{-1}$;

d) when $\beta^2 > 4\alpha D$ and $(2\alpha V_0 + \beta)^2 > (\beta^2 - 4\alpha D)$

$$y_4 = \int_{\mathbf{V}_i}^{\mathbf{V}_0}\frac{d\mathbf{V}_0(\mathbf{V}_m - \mathbf{V}_0)}{\alpha V_0^2 + \beta V_0 + D} =$$

$$= \omega\,\operatorname{arth}\frac{2\alpha N(\mathbf{V}_0 - \mathbf{V}_i)}{(2\alpha\mathbf{V}_0 + \beta)(2\alpha\mathbf{V}_i + \beta) - N^2} - \frac{1}{2\alpha}\ln M_2; \quad (20)$$

e) when $\beta^2 - 4\alpha D = 0$

$$y_5 = \int_{\mathbf{V}_i}^{\mathbf{V}_0}\frac{d\mathbf{V}_0(\mathbf{V}_m - \mathbf{V}_0)}{\alpha\mathbf{V}_0^2 + \beta\mathbf{V}_0 + D} =$$

$$= \frac{(2\mathbf{V}_m + \beta)(\mathbf{V}_i - \mathbf{V}_0)}{\alpha(\mathbf{V}_0 + 0.5E)(\mathbf{V}_i + 0.5E)} - \frac{1}{2\alpha}\ln M_2. \qquad (21)$$

The derived equations (12) and (17)–(21) make it possible by the method of successive integration to calculate the trajectory of particle motion in a non-uniform moving medium.

The figure shows the entrainment (deceleration) path of the particles by the medium as a function of the parameter L calculated from Eq. (12) (curve 1) and of the parameter D calculated from (18) (curve 2).

NOTATION

m is the particle mass, equal to $(\pi d^3/6)\rho_2$; d is the particle diameter, m; ρ_1 and ρ_2 are the densities of the medium and particle, kg/m^3; ψ is the coefficient of hydraulic resistance of the medium; S is the mid-cross-sectional area of the particle, equal to $\pi d^2/4$, m^2; V_m, V_p, and $\mathbf{V}_0$ are the vectors of velocity of the medium, the particle, and the relative velocity of the particle (related to the medium), m/sec; ν and μ are the medium viscosities, m^2/sec, N · sec/m^2; t_0, t_1, t_2 are the times, sec; F_e is the vector applied to the external force particle; m_1 is the mass of fluid displaced by a particle equal to $\pi d^3\rho_1/6$, kg.

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Institute of Industrial and Hygienic Purification of Gases, Moscow